Power Series

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A **power series** about a number x_0 is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + ...$$

where c_n are constants and x is a variable. When we plug in a specific value for x, the corresponding series may be convergent or divergent. An interesting question to ask is to find the set of values for x so that the given power series converges.

Theorem 1. *For a given power series* $\sum_{n=0}^{\infty} c_n(x-x_0)^n$*, one of the following will happen:*

1. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ *converges only at* $x = x_0$.
2. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ *converges for all* $x \in \mathbb{R}$.
3. *There exists a constant* $R > 0$ *such that* $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ *converges for* $|x-x_0| < R$ *and diverges for* $|x-x_0| > R$.

Definition 2. The number R appeared in the above theorem is called the **radius of convergence** of the series $\sum_{n=0}^{\infty} c_n(x-x_0)^n$. If the series converges only at $x = x_0$, we define $R = 0$; if the series converges for all $x \in \mathbb{R}$, we define $R = \infty$.

Remark 3. For points on the radius of convergence, the above theorem does not provide a conclusive answer about the convergence of the power series at these points.

Theorem 4. $\sum_{n=0}^{\infty} c_n(x-x_0)^n$ *is a power series. The radius of convergence of this power series is:*

$$R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}}$$

provided the above limits converges or diverge to ∞.

Proof. First observe that

$$\lim_{n\to\infty} \frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n} = |x-x_0| \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$

If $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = 0$, then for any $x \neq x_0$,

$$\lim_{n\to\infty} \frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n} = |x-x_0| \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \infty$$

So by the Ratio Test the power series diverges for all $x \neq x_0$, the radius of convergence is $R = 0$.

If $\lim_{n\to\infty} \frac{c_n}{c_{n+1}}$ is a positive constant, then

$$\lim_{n\to\infty} \frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n} = \frac{|x-x_0|}{\lim_{n\to\infty} \frac{c_n}{c_{n+1}}} = \begin{cases} < 1 \text{ if } |x-x_0| < \lim_{n\to\infty} \frac{c_n}{c_{n+1}} \\ > 1 \text{ if } |x-x_0| > \lim_{n\to\infty} \frac{c_n}{c_{n+1}} \end{cases}$$

By the Ratio Test, we see the radius of convergence is $R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}}$.

If $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = \infty$, then for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n} = |x-x_0| \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0$$

So by the Ratio Test the power series converges for all $x \in \mathbb{R}$, the radius of convergence is $R = \infty$.

Next observe that

$$\lim_{n\to\infty} \sqrt[n]{|c_n(x-x_0)^n|} = |x-x_0| \lim_{n\to\infty} \sqrt[n]{c_n}$$

If $\lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}} = 0$, then for any $x \neq x_0$,

$$\lim_{n\to\infty} \sqrt[n]{|c_n(x-x_0)^n|} = |x-x_0| \lim_{n\to\infty} \sqrt[n]{c_n} = \infty$$

So by the Root Test the power series diverges for all $x \neq x_0$, the radius of convergence is $R = 0$.

If $\lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}}$ is a positive constant, then

$$\lim_{n\to\infty} \sqrt[n]{|c_n(x-x_0)^n|} = \frac{|x-x_0|}{\lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}}} = \begin{cases} < 1 \text{ if } |x-x_0| < \lim_{n\to\infty} \frac{c_n}{c_{n+1}} \\ > 1 \text{ if } |x-x_0| > \lim_{n\to\infty} \frac{c_n}{c_{n+1}} \end{cases}$$

By the Root Test, we see the radius of convergence is $R = \lim_{n\to\infty} \frac{c_n}{c_{n+1}}$.

If $\lim_{n\to\infty} \frac{1}{\sqrt[n]{c_n}} = \infty$, then for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \sqrt[n]{|c_n(x-x_0)^n|} = |x-x_0| \lim_{n\to\infty} \sqrt[n]{c_n} = 0$$

So by the Root Test the power series converges for all $x \in \mathbb{R}$, the radius of convergence is $R = \infty$.

$\square$

Example 5. *Determine for which choice of x is the power series $\displaystyle\sum_{n=0}^{\infty} x^n$ convergent.*

The radius of convergence is

$$R = \lim_{n\to\infty} |\frac{c_n}{c_{n+1}}| = \lim_{n\to\infty} 1 = 1$$

So the series converges for $|x| < 1$ and diverges for $|x| > 1$.

We need to study the case $|x| = 1$ separately. When $|x| = 1$, the limit $\lim\limits_{n\to\infty} |x|^n = \lim\limits_{n\to\infty} 1 = 1 \neq 0$, so the series diverges at $x = \pm 1$.

We conclude the power series converges for $|x| < 1$ and diverges for $|x| \geq 1$.

Example 6. *Find the radius of convergence of the power series* $\sum_{n=0}^{\infty} \frac{(-3x)^n}{\sqrt{n+1}}$ *and discuss the convergence for* x *at the radius of convergence.*

The power series is $\sum_{n=0}^{\infty} \frac{(-3x)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}} x^n$

The radius of convergence is

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-3)^n}{\sqrt{n+1}}}{\frac{(-3)^{n+1}}{\sqrt{n+2}}}\right| = \frac{1}{3} \lim_{n\to\infty} \sqrt{\frac{n+2}{n+1}} = \frac{1}{3}$$

When $x = \frac{1}{3}$*, the series becomes* $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$*, which is convergent by the Alternating Convergence Test.*

When $x = -\frac{1}{3}$*, the series becomes* $\sum_{n=0}^{\infty} \frac{1}{n+1}$*, which is divergent.*

Example 7. *The Bessel function is defined as*

$$J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

Find its domain.

To find its domain is the same as to find all x *at which the power series converges. Let* $y = x^2$*, then consider*

$$\tilde{J}(y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{2^{2n}(n!)^2}$$

We see $J(x) = \tilde{J}(x^2)$*, so we first figure out the domain for* $\tilde{J}$*.*

The radius of convergence for $\tilde{J}$ *is*

$$R = \lim_{n\to\infty} \left|\frac{\frac{(-1)^n}{2^{2n}(n!)^2}}{\frac{(-1)^{n+1}}{2^{2(n+1)}((n+1)!)^2}}\right| = \lim_{n\to\infty} 4(n+1)^2 = \infty$$

So the domain of $\tilde{J}$ *is* $\mathbb{R}$*, hence the domain of* $J(x)$ *is* $\mathbb{R}$*.*